Project systems theory – Solutions

Final exam 2017–2018, Thursday 25 January 2018, 9:00 – 12:00

Problem 1

(3 + 3 + 8 = 14 points)

A simple model of a magnetic levitation system is given as

$$m\ddot{q}(t) = mg - \frac{1}{2}\frac{L}{(1+q(t))^2}u^2(t), \tag{1}$$

with $q(t)$ the position of the levitated mass with mass $m > 0$ and $g > 0$ the gravitational constant. The input current to the electromagnet that suspends the mass is denoted by $u(t)$ and $L > 0$ is a constant.

(a) To write (1) in nonlinear state-space form, introduce the state

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}. \tag{2}$$

Then, it is immediate that $\dot{x}_1 = x_2$. The dynamics for x_2 follows from (1), leading to

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ g - \frac{1}{2m}\frac{L}{(1+x_1)^2}u^2 \end{bmatrix} = f(x, u). \tag{3}$$

(b) Let

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \bar{q} \\ 0 \end{bmatrix} \tag{4}$$

be the desired equilibrium point for some $\bar{q} > 0$. To find the constant input $u(t) = \bar{u}$, the equation

$$0 = f(\bar{x}, \bar{u}) \tag{5}$$

needs to be solved. Using (3), we obtain $0 = \bar{x}_2$ for the first coordinate, which indeed corresponds to (4). The equation for the second coordinate yields

$$g = \frac{1}{2m}\frac{L}{(1+\bar{q})^2}\bar{u}^2, \tag{6}$$

which has the solution (recall that a positive solution $\bar{u} > 0$ is sought)

$$\bar{u} = \sqrt{\frac{2mg}{L}}\big(1 + \bar{q}\big). \tag{7}$$

(c) In order to find the linearized dynamics around the equilibrium point given by $\bar{x}$ and $\bar{u}$, define the perturbations

$$\tilde{x} = x - \bar{x}, \qquad \tilde{u} = u - \bar{u}. \tag{8}$$

Then, the linearized dynamics is given as

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\tilde{x}(t) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\tilde{u}(t), \tag{9}$$

after which it can be concluded from (3) that

$$\frac{\partial f}{\partial x}(x,u) = \begin{bmatrix} 0 & 1 \\ \frac{L}{m}(1+x_1)^{-3}u^2 & 0 \end{bmatrix}. \tag{10}$$

Evaluation of the result at $(\bar{x},\bar{u})$ gives, after substitution of (7),

$$\frac{\partial f}{\partial x}(\bar{x},\bar{u}) = \begin{bmatrix} 0 & 1 \\ 2g(1+\bar{q})^{-1} & 0 \end{bmatrix}. \tag{11}$$

Similarly,

$$\frac{\partial f}{\partial u}(x,u) = \begin{bmatrix} 0 \\ -\frac{1}{m}\frac{L}{(1+x_1)^2}u \end{bmatrix} \tag{12}$$

such that

$$\frac{\partial f}{\partial u}(\bar{x},\bar{u}) = \begin{bmatrix} 0 \\ -\sqrt{\frac{2gL}{m}}\frac{1}{1+\bar{q}} \end{bmatrix}. \tag{13}$$

Problem 2 (14 points)

Consider the polynomial

$$p(\lambda) = \lambda^4 + 3\lambda^3 + \lambda^2 + a\lambda + 2a \tag{14}$$

where $a \in \mathbb{R}$.

As a first step in finding the values of a for which (14) is stable, note that a necessary condition for stability is that all coefficients have the same sign. As a result, we need $a > 0$.

To proceed, consider the following Routh-Hurwitz table:

	λ^4	λ^3	λ^2	λ^1	λ^0	
$3\times$	1	3	1	a	$2a$	
$1\times$	3		a			
$(3-a)\times$		9	$3-a$	$3a$	$6a$	(result of step 1)
$9\times$		$3-a$		$6a$		
			$(3-a)^2$	$-3a(a+15)$	$6a(3-a)$	(result of step 2)

Recall that the Routh-Hurwitz criterion states that the polynomial p is stable if its two leading coefficients have the same sign and the polynomial obtained in step 1 is stable. The leading coefficients 1 and 3 satisfy the first criterion. Then, in order for the polynomial obtained in step 1 to be stable, it is again required that all coefficients have the same sign. This leads to the condition $0 < a < 3$.

The repetition of this reasoning (note that the sign of 9 and $3 - a$ is the same as $0 < a < 3$) leads to the result of step 2. Again, for the polynomial obtained in step 2, a necessary condition for stability is that all coefficients have the same sign. In this case, we need

$$-3a(a + 15) > 0, \tag{15}$$

which however contradicts with the earlier condition $0 < a < 3$ (explicitly solving for (15) gives $-15 < a < 0$). Consequently, there does not exist a parameter $a \in \mathbb{R}$ such that the polynomial (14) is stable.

Problem 3 (4 + 12 + 6 = 22 points)

Consider the system

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t), \tag{16}$$

with state $x(t) \in \mathbb{R}^2$, output $y(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} -6 & 3 \\ -7 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} -2 & 1 \end{bmatrix}. \tag{17}$$

(a) The observability matrix of the pair (A, C) is given by

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}. \tag{18}$$

Then, as the rows in this matrix are linearly independent, it follows that

$$\operatorname{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix} = 2. \tag{19}$$

Hence, the system (17) is observable.

(b) To find a nonsingular matrix T and real numbers a_1, a_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix}, \qquad CT^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{20}$$

holds, the duality between controllability and observability can be exploited. Namely, observability of (A, C) implies that $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is controllable. Thus, we consider the system

$$A^{\mathrm{T}} = \begin{bmatrix} -6 & -7 \\ 3 & 4 \end{bmatrix}, \qquad C^{\mathrm{T}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \tag{21}$$

and look for a transformation S such that

$$S^{-1}A^{\mathrm{T}}S = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \qquad S^{-1}C^{\mathrm{T}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{22}$$

Then, using

$$\begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix} = \left(S^{-1}A^{\mathrm{T}}S\right)^{\mathrm{T}} = S^{\mathrm{T}}AS^{-\mathrm{T}}, \qquad \begin{bmatrix} 0 & 1 \end{bmatrix} = \left(S^{-1}C^{\mathrm{T}}\right)^{\mathrm{T}} = CS^{-\mathrm{T}}, \tag{23}$$

it is verified that the desired transformation T is given by

$$T = S^{\mathrm{T}}. \tag{24}$$

In the remainder of this problem, a matrix S satisfying (22) will be constructed. Note that

$$\begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix} \tag{25}$$

is in companion form, such that a_1 and a_2 are the coefficients of the characteristic polynomial of A^{T}. The characteristic polynomial reads

$$\Delta_{A^{\mathrm{T}}}(\lambda) = \det(\lambda I - A^{\mathrm{T}}) = \begin{vmatrix} \lambda + 6 & 7 \\ -3 & \lambda - 4 \end{vmatrix} = (\lambda + 6)(\lambda - 4) + 21 = \lambda^2 + 2\lambda - 3, \tag{26}$$

such that

$$a_1 = 2, \qquad a_2 = -3. \tag{27}$$

Observe that $\Delta_{A^{\mathrm{T}}}(\lambda) = \Delta_A(\lambda)$. To find the corresponding transformation S, compute

$$q_2 = C^{\mathrm{T}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \tag{28}$$

$$q_1 = A^{\mathrm{T}}C^{\mathrm{T}} + a_1 C^{\mathrm{T}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{29}$$

after which the transformation matrix S is given as

$$S = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}. \tag{30}$$

Using the inverse

$$S^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \tag{31}$$

it is readily verified that (22) holds. Finally, the relation (24) gives the desired transformation

$$T = S^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \tag{32}$$

(c) By similarity transformation, the matrix $A - GC$ and

$$T(A - GC)T^{-1} = TAT^{-1} - TGCT^{-1} \tag{33}$$

have the same eigenvalues. Denoting $\bar{G} = TG$ and defining the resulting matrix as

$$\bar{G} = \begin{bmatrix} g_2 \\ g_1 \end{bmatrix}, \tag{34}$$

it follows that

$$TAT^{-1} - \bar{G}CT^{-1} = \begin{bmatrix} 0 & -(a_2 + g_2) \\ 1 & -(a_1 + g_1) \end{bmatrix}. \tag{35}$$

Here, the result (20) is used. As the matrix in (35) is in companion form (in fact, the transpose of a companion form), it follows that its characteristic polynomial reads

$$\Delta_{T(A-GC)T^{-1}}(\lambda) = \lambda^2 + (a_1 + g_1)\lambda + (a_2 + g_2). \tag{36}$$

To place the eigenvalues at -2 and -4, this polynomial should have these values as its roots, i.e.,

$$\lambda^2 + (a_1 + g_1)\lambda + (a_2 + g_2) = (\lambda + 2)(\lambda + 4) = \lambda^2 + 6\lambda + 8, \tag{37}$$

leading to

$$g_1 = 6 - a_1 = 6 - 2 = 4, \qquad g_2 = 8 - a_2 = 8 + 3 = 11. \tag{38}$$

Then, using the definition $\bar{G} = TG$ and (34), it follows that

$$G = T^{-1}\bar{G} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 11 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 26 \end{bmatrix}. \tag{39}$$

Problem 4 (4 + 4 + 7 + 7 = 22 points)

Consider the system

$$\dot{x}(t) = \begin{bmatrix} 8 & 2 & -7 \\ -12 & -2 & 14 \\ 0 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t). \tag{40}$$

(a) Stability of (40) is determined by the eigenvalues of A. Due to the upper block-triangular structure of A, its spectrum equals

$$\sigma(A) = \sigma\left(\begin{bmatrix} 8 & 2 \\ -12 & -2 \end{bmatrix}\right) \cup \{-3\}. \tag{41}$$

The eigenvalues of the upper-left block are obtained as the roots of

$$\begin{vmatrix} \lambda - 8 & -2 \\ 12 & \lambda + 2 \end{vmatrix} = (\lambda - 8)(\lambda + 2) + 24 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4), \tag{42}$$

which read 2 and 4. Thus, the spectrum of A reads

$$\sigma(A) = \{-3, 2, 4\} \tag{43}$$

and the system is not (asymptotically stable). Namely, there exist eigenvalues with positive real parts.

(b) The reachable subspace $\mathcal{W}$ of the system (40) is given as

$$\mathcal{W} = \operatorname{im}\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \operatorname{im}\begin{bmatrix} 0 & -5 & 5 \\ 1 & 12 & -6 \\ 1 & -3 & 9 \end{bmatrix}, \tag{44}$$

$$= \operatorname{im}\begin{bmatrix} 0 & -5 & 5 \\ 1 & 15 & -6 \\ 1 & 0 & 9 \end{bmatrix} = \operatorname{im}\begin{bmatrix} 0 & -1 & 5 \\ 1 & 3 & -6 \\ 1 & 0 & 9 \end{bmatrix}, \tag{45}$$

where elementary column operations are performed to obtain the results (45). Now, as

$$\begin{bmatrix} 5 \\ -6 \\ 9 \end{bmatrix} = 9\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 5\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \tag{46}$$

it follows that the dimension of the reachable subspace $\mathcal{W}$ is two. Thus, a basis representation is given as

$$\mathcal{W} = \operatorname{span}\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}\}. \tag{47}$$

For the remainder of this problem, consider the system

$$\dot{x}(t) = \begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & b & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ b \end{bmatrix} u(t), \tag{48}$$

where a and b are real parameters.

(c) Using the Hautus test, the system (48) is controllable if and only if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \tag{49}$$

for all $\lambda \in \sigma(A)$. Here, $n = 3$.

Due to the lower triangular structure of A, it is immediate that

$$\sigma(A) = \{a, -1\}, \tag{50}$$

where the eigenvalue a has multiplicity 2. For $\lambda = a$, (49) reads

$$\begin{bmatrix} A - aI & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & b & -1-a & b \end{bmatrix}, \tag{51}$$

whereas $\lambda = -1$ leads to

$$\begin{bmatrix} A + I & B \end{bmatrix} = \begin{bmatrix} a+1 & 0 & 0 & 1 \\ 1 & a+1 & 0 & 0 \\ 0 & b & 0 & b \end{bmatrix}, \tag{52}$$

For the matrix in (52) to have full column rank n, it is necessary that $b \neq 0$. In this case (52) has full column rank for all a. Returning to (51), it is clear that the condition $b \neq 0$ implies full column rank for all a. Thus, the system (48) is controllable if and only if

$$a \in \mathbb{R}, \quad b \neq 0. \tag{53}$$

(d) The system (48) is stabilizable if and only if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \tag{54}$$

for all $\lambda \in \sigma(A)$ such that $\text{Re}(\lambda) \geq 0$, as follows from the Hautus test. As before, $n = 3$.

Considering the spectrum of A in (50), two cases can be considered. First, for $a < 0$, the system is asymptotically stable and, hence, stabilizable. Thus, $a < 0$ is sufficient for stabilizability. Next, we consider the case $a \geq 0$. Then, the matrix in (51) needs to have full column rank, which is implied by the condition $a \geq 0$ and the system is stabilizable.

Combining the two cases above, it is concluded that (48) is stabilizable for all

$$a \in \mathbb{R}, \quad b \in \mathbb{R}. \tag{55}$$

Finally, note that the condition for controllability (53) implies stabilizability as expected.

Problem 5 (4 + 14 = 18 points)

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{56}$$

and recall the general solution for a given initial condition x_0 and input $u(\cdot)$ as

$$x_u(t, x_0) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s. \tag{57}$$

Assume that the system (56) is controllable and define the input signal

$$\bar{u}(t) = B^{\mathrm{T}}e^{-A^{\mathrm{T}}t}K^{-1}e^{-AT}x_T \tag{58}$$

for some $x_T \in \mathbb{R}^n$ and fixed $T > 0$ and where

$$K = \int_0^T e^{-As}BB^{\mathrm{T}}e^{-A^{\mathrm{T}}s}\,\mathrm{d}s. \tag{59}$$

(a) Let $x_0 = 0$. Then, a direct computation shows

$$\begin{aligned}
x_{\bar{u}}(T, 0) &= \int_0^T e^{A(T-s)}B\bar{u}(s)\,\mathrm{d}s & (60)\\
&= \int_0^T e^{A(T-s)}BB^{\mathrm{T}}e^{-A^{\mathrm{T}}s}K^{-1}e^{-AT}x_T\,\mathrm{d}s & (61)\\
&= e^{AT}\int_0^T e^{-As}BB^{\mathrm{T}}e^{-A^{\mathrm{T}}s}\,\mathrm{d}s\,K^{-1}e^{-AT}x_T & (62)\\
&= e^{AT}KK^{-1}e^{-AT}x_T & (63)\\
&= e^{AT}e^{-AT}x_T & (64)\\
&= x_T. & (65)
\end{aligned}$$

Here, (61) follows by direct substitution of (58), whereas the definition of K in (59) is used to obtain (63).

(b) Nonsingularity of K will be shown by contradiction. To this end, assume that K is singular. Then, there exists $v \in \mathbb{R}$, $v \neq 0$, such that $Kv = 0$. As a result, also $v^{\mathrm{T}}Kv = 0$, i.e.,

$$0 = \int_0^T v^{\mathrm{T}}e^{-As}BB^{\mathrm{T}}e^{-A^{\mathrm{T}}s}v\,\mathrm{d}s = \int_0^T \left\|B^{\mathrm{T}}e^{-A^{\mathrm{T}}s}v\right\|_2^2\mathrm{d}s, \tag{66}$$

where $\|\cdot\|_2$ denotes the Euclidian norm. As $B^{\mathrm{T}}e^{-A^{\mathrm{T}}s}v$ is a continuous function of s, (66) is equivalent to

$$B^{\mathrm{T}}e^{-A^{\mathrm{T}}s}v = 0, \quad \forall s \in [0, T]. \tag{67}$$

In the remainder, we will consider the transposed version

$$v^{\mathrm{T}}e^{-As}B = 0, \quad \forall s \in [0, T]. \tag{68}$$

Evaluating (68) for $s = 0$ leads to

$$v^{\mathrm{T}}B = 0. \tag{69}$$

Next, as $v^{\mathrm{T}}e^{-As}B$ is identically zero over the interval $[0, T]$, it follows that

$$\tfrac{\mathrm{d}^k}{\mathrm{d}s^k}\left\{v^{\mathrm{T}}e^{-As}B\right\} = (-1)^k v^{\mathrm{T}}A^k e^{-As}B = 0 \tag{70}$$

for all $s \in [0, T]$. Evaluation of (70) for $s = 0$ gives

$$v^{\mathrm{T}} A^k B = 0, \quad k = 1, 2, \ldots, \tag{71}$$

which in turn implies

$$v^{\mathrm{T}} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0, \tag{72}$$

where it is recalled that $v \neq 0$. Thus,

$$\operatorname{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} < n, \tag{73}$$

which contradicts the fact that (56) is controllable. As a result, K is nonsingular.

(10 points free)